

## COMPLEX SPACE FORMS IMMERSED IN COMPLEX SPACE FORMS

BY

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**ABSTRACT.** We determine all the isometric immersions of complex space forms into complex space forms. Our result can be considered as the local version of a well-known result of Calabi.

A Kaehler manifold of constant holomorphic curvature is called a *complex space form*. By a *Kaehler submanifold* we mean a complex submanifold with the induced Kaehler metric. E. Calabi [1] gave a classification of Kaehler imbeddings of complete and simply connected complex space forms into complete and simply connected complex space forms. The local version of Calabi's result has been conjectured to be true by the second author [4] and he gave some partial solutions [2], [3].

The purpose of this paper is to prove the following two theorems which furnish the complete solutions to the conjecture. Throughout this paper we denote by  $M_n(c)$  an  $n$ -dimensional complex space form of constant holomorphic curvature  $c$ .

**THEOREM 1.** *Let  $M_n(c)$  be a Kaehler submanifold immersed in  $M_{n+p}(\tilde{c})$ . If  $\tilde{c} > 0$  and the immersion is full, then  $\tilde{c} = \nu c$  and  $n + p = \binom{n+\nu}{\nu} - 1$  for some positive integer  $\nu$ .*

**THEOREM 2.** *Let  $M_n(c)$  be a Kaehler submanifold immersed in  $M_{n+p}(\tilde{c})$ . If  $\tilde{c} \leq 0$ , then  $\tilde{c} = c$  (i.e.,  $M_n(c)$  is totally geodesic in  $M_{n+p}(\tilde{c})$ ).*

1. **Kaehler submanifolds in  $M_{n+p}(\tilde{c})$ .** Let  $M$  be an  $n$ -dimensional Kaehler submanifold immersed in  $M_{n+p}(\tilde{c})$ . We choose a local field of unitary frames  $e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}$  in  $M_{n+p}(\tilde{c})$  in such a way that, restricted to  $M$ ,

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<sup>(2)</sup>Throughout this paper we use the following convention on the range of indices unless otherwise stated:

$A, B, C, \dots = 1, \dots, n, n+1, \dots, n+p,$

$i, j, k, \dots = 1, \dots, n,$

$\alpha, \beta, \gamma, \dots = n+1, \dots, n+p.$

$e_1, \dots, e_n$  are tangent to  $M$ . Let  $\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^{n+p}$  be the field of dual frames. Then the Kaehler metric  $\tilde{g}$  of  $M_{n+p}(\tilde{c})$  is given by  $\tilde{g} = \Sigma_A \omega^A \bar{\omega}^A$  and the structure equations of  $M_{n+p}(\tilde{c})$  are given by  $(^2)$

$$(1.1) \quad d\omega^A + \sum_B \omega_B^A \wedge \omega^B = 0, \quad \omega_B^A + \bar{\omega}_A^B = 0,$$

$$(1.2) \quad d\omega_B^A + \sum_C \omega_C^A \wedge \omega_B^C = \tilde{\Omega}_B^A, \quad \tilde{\Omega}_B^A = \sum_{C,D} \tilde{R}_{BCD}^A \omega^C \wedge \bar{\omega}^D.$$

Since  $M_{n+p}(\tilde{c})$  is a complex space form of constant holomorphic curvature  $\tilde{c}$ , we have

$$(1.3) \quad \tilde{R}_{BCD}^A = \frac{\tilde{c}}{4} (\delta_B^A \delta_{CD} + \delta_C^A \delta_{BD}).$$

Restricting these forms to  $M$ , we have

$$(1.4) \quad \omega^\alpha = 0,$$

and the Kaehler metric  $g$  of  $M$  is given by  $g = \Sigma_i \omega^i \bar{\omega}^i$ .

It follows from (1.1), (1.4) and Cartan's lemma that we may write

$$(1.5) \quad \omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

Moreover we obtain

$$(1.6) \quad d\omega^i + \sum_j \omega_j^i \wedge \omega^j = 0, \quad \omega_j^i + \bar{\omega}_i^j = 0,$$

$$(1.7) \quad d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k = \Omega_j^i, \quad \Omega_j^i = \sum_{k,l} R_{jk\bar{l}}^i \omega^k \wedge \bar{\omega}^l,$$

$$(1.8) \quad d\omega_\beta^\alpha + \sum_\gamma \omega_\gamma^\alpha \wedge \omega_\beta^\gamma = \Omega_\beta^\alpha, \quad \Omega_\beta^\alpha = \sum_{k,l} R_{\beta k\bar{l}}^\alpha \omega^k \wedge \bar{\omega}^l.$$

From (1.2), (1.3), (1.4), (1.5), and (1.7), we have

$$(1.9) \quad \Omega_j^i = \sum_{k,l} \left\{ \frac{\tilde{c}}{4} (\delta_j^i \delta_{kl} + \delta_k^i \delta_{jl}) - \sum_\alpha h_{jk}^\alpha \bar{h}_{il}^\alpha \right\} \omega^k \wedge \bar{\omega}^l.$$

Similarly from (1.2), (1.3), (1.4), (1.5), and (1.8), we have

$$(1.10) \quad \Omega_\beta^\alpha = \sum_{k,l} \left( \frac{\tilde{c}}{4} \delta_\beta^\alpha \delta_{kl} + \sum_j h_{jk}^\alpha \bar{h}_{jl}^\beta \right) \omega^k \wedge \bar{\omega}^l.$$

If we define  $h_{ijk}^\alpha$  and  $h_{ij\bar{k}}^\alpha$  by

$$(1.11) \quad \sum_k h_{ijk}^\alpha \omega^k + \sum_k h_{ij\bar{k}}^\alpha \bar{\omega}^k = dh_{ij}^\alpha - \sum_k h_{kj}^\alpha \omega_i^k - \sum_k h_{ik}^\alpha \omega_j^k + \sum_\beta h_{ij}^\beta \omega_\beta^\alpha,$$

then we can easily see

$$(1.12) \quad h_{ijk}^\alpha = h_{ikj}^\alpha, \quad h_{ij\bar{k}}^\alpha = 0.$$

We may define inductively the successive derivatives of  $h_{ij}^\alpha$  by

$$(1.13) \quad \left\{ \begin{aligned} & \sum_l h_{i_1 \dots i_k l}^\alpha \omega^l + \sum_l h_{i_1 \dots i_k \bar{l}}^\alpha \bar{\omega}^l \\ &= dh_{i_1 \dots i_k}^\alpha - \sum_{r=1}^k \sum_{j_r} h_{i_1 \dots j_r \dots i_k}^\alpha \omega_{j_r}^{j_r} + \sum_\beta h_{i_1 \dots i_k}^\beta \omega_\beta^\alpha, \\ & \sum_m h_{i_1 \dots i_k l m}^\alpha \omega^m + \sum_m h_{i_1 \dots i_k \bar{l} m}^\alpha \bar{\omega}^m \\ &= dh_{i_1 \dots i_k \bar{l}}^\alpha - \sum_{r=1}^k \sum_{j_r} h_{i_1 \dots j_r \dots i_k \bar{l}}^\alpha \omega_{j_r}^{j_r} \\ & \quad - \sum_m h_{i_1 \dots i_k \bar{m}}^\alpha \bar{\omega}_l^m + \sum_\beta h_{i_1 \dots i_k \bar{l}}^\beta \omega_\beta^\alpha. \end{aligned} \right.$$

If  $h_{i_1 \dots i_k}^\alpha$  is symmetric with respect to all indices, then taking the exterior derivative of the first equality of (1.13) and using (1.6), (1.7), and (1.8), we get

$$\begin{aligned} & \sum_{l,m} h_{i_1 \dots i_k l m}^\alpha \omega^l \wedge \omega^m - \sum_{l,m} (h_{i_1 \dots i_k l m}^\alpha - h_{i_1 \dots i_k \bar{m} l}^\alpha) \omega^l \wedge \bar{\omega}^m \\ & \quad + \sum_{l,m} h_{i_1 \dots i_k \bar{l} m}^\alpha \bar{\omega}^l \wedge \omega^m \\ &= - \sum_{r=1}^k \sum_{j_r} h_{i_1 \dots j_r \dots i_k}^\alpha \Omega_{j_r}^{j_r} + \sum_\beta h_{i_1 \dots i_k}^\beta \Omega_\beta^\alpha, \end{aligned}$$

from which, together with (1.9) and (1.10), we obtain

$$(1.14) \quad h_{i_1 \dots i_k l m}^\alpha = h_{i_1 \dots i_k m l}^\alpha, \quad h_{i_1 \dots i_k l \bar{m}}^\alpha = h_{i_1 \dots i_k \bar{m} l}^\alpha,$$

$$(1.15) \quad \begin{aligned} & h_{i_1 \dots i_k l \bar{m}}^\alpha - h_{i_1 \dots i_k \bar{m} l}^\alpha \\ &= \frac{\tilde{c}}{4} \left\{ (k-1) h_{i_1 \dots i_k}^\alpha \delta_{lm} + \sum_{r=1}^k h_{i_1 \dots l \dots i_k}^\alpha \delta_{i_r m} \right\} \\ & \quad - \sum_{r=1}^k \sum_{j_r, \beta} h_{i_1 \dots j_r \dots i_k}^\alpha h_{i_r e}^\beta \bar{h}_{j_r m}^\beta - \sum_{j, \beta} h_{i_1 \dots i_k}^\beta h_{j l}^\alpha \bar{h}_{j m}^\beta. \end{aligned}$$

Since  $h_{ij}^\alpha$  and  $h_{ij}^\alpha$  are symmetric with respect to all indices, (1.14) implies inductively that

$$(1.16) \quad h_{i_1 \dots i_k}^\alpha \text{ is symmetric with respect to all indices,}$$

and consequently (1.15) holds for all integers  $k \geq 2$ .

2. **Proof of theorems.** Since  $M_n(c)$  is a complex space form of constant holomorphic curvature  $c$ , the curvature forms  $\Omega_j^i$  are given by

$$\Omega_j^i = \sum_{k,l} R_{jk\bar{l}}^i \omega^k \wedge \bar{\omega}^l = \frac{c}{4} \sum_{k,l} (\delta_j^i \delta_{kl} + \delta_k^i \delta_{jl}) \omega^k \wedge \bar{\omega}^l,$$

which, together with (1.9), implies

$$(2.1) \quad \sum_{\alpha} h_{ik}^{\alpha} \bar{h}_{jl}^{\alpha} = \frac{\tilde{c} - c}{4} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}).$$

From (2.1) and the second property of (1.12), we obtain

$$(2.2) \quad \sum_{\alpha} h_{i_1 \dots i_k}^{\alpha} \bar{h}_{jl}^{\alpha} = 0 \quad \text{for } k \geq 3.$$

Substituting (2.1) and (2.2) into (1.15), we have

$$\begin{aligned} h_{i_1 \dots i_k l \bar{m}}^{\alpha} &= h_{i_1 \dots i_k \bar{m} l}^{\alpha} - \frac{\tilde{c} - kc}{4} h_{i_1 \dots i_k}^{\alpha} \delta_{lm} \\ &\quad + \frac{c}{4} \sum_{r=1}^k h_{i_1 \dots i_r \dots i_k}^{\alpha} \delta_{l_r m} \quad \text{for } k \geq 3, \end{aligned}$$

and, in particular, applying this relation to the first term of the right-hand side repeatedly and taking account of (1.12), we can obtain

$$(2.3) \quad h_{i_1 \dots i_k l}^{\alpha} = -\frac{\tilde{c} - (k-1)c}{4} \sum_{r=1}^k h_{i_1 \dots \hat{i}_r \dots i_k}^{\alpha} \delta_{l_r l},$$

where the notation  $\hat{\phantom{x}}$  means the omission of the index  $i_r$ . Since  $h_{ij}^{\alpha}$  is symmetric, from (2.1), (2.2), and (2.3), we have

$$(2.4) \quad \begin{aligned} h_{ijk}^{\alpha} \bar{h}_{lmn}^{\alpha} &= \frac{(\tilde{c} - c)(\tilde{c} - 2c)}{4^2} (\delta_{il} \delta_{jm} \delta_{kn} + \delta_{il} \delta_{jn} \delta_{km} + \delta_{im} \delta_{jn} \delta_{kl} \\ &\quad + \delta_{im} \delta_{jl} \delta_{kn} + \delta_{in} \delta_{jl} \delta_{km} + \delta_{in} \delta_{jm} \delta_{kl}). \end{aligned}$$

First of all, as a generalization of (2.1) and (2.4), we shall prove the following.

LEMMA 1.

$$(2.5) \quad \sum_{\alpha} h_{i_1 \dots i_k}^{\alpha} \bar{h}_{j_1 \dots j_l}^{\alpha} = \begin{cases} 0 & \text{for } k \neq l, \\ \frac{1}{4^{k-1}} \prod_{r=1}^{k-1} (\tilde{c} - rc) \sum_{\sigma} \delta_{\sigma(i_1)j_1} \cdots \delta_{\sigma(i_k)j_k} & \text{for } k = l, \end{cases}$$

where  $\Sigma_\sigma$  is the summation on all permutations with respect to indices  $i_1, \dots, i_k$ .

PROOF. We shall prove the second equality by induction. The cases where  $k = 2$ , and  $k = 3$  reduce to (2.1) and (2.4) respectively.

We suppose that the following equalities hold:

$$(2.6) \quad \sum_{\alpha} h_{i_1 \dots i_l}^{\alpha} \bar{h}_{j_1 \dots j_l}^{\alpha} = \frac{1}{4^{l-1}} \prod_{r=1}^{l-1} (\tilde{c} - rc) \sum_{\sigma} \delta_{\sigma(i_1)j_1} \cdots \delta_{\sigma(i_l)j_l} \quad \text{for } l \leq k.$$

Then it follows from (2.3) that

$$\sum_{\alpha} h_{i_1 \dots i_k}^{\alpha} \bar{h}_{j_1 \dots j_k m}^{\alpha} = - \sum_{\alpha} h_{i_1 \dots i_k m}^{\alpha} \bar{h}_{j_1 \dots j_k}^{\alpha} \equiv 0 \pmod{\sum_{\alpha} h_{i_1 \dots i_{k-1}}^{\alpha} \bar{h}_{m_1 \dots m_k}^{\alpha} \delta_{ij}}.$$

Making use of (2.3) and the supposition (2.6) of the induction repeatedly, we see that

$$(2.7) \quad \sum_{\alpha} h_{i_1 \dots i_k}^{\alpha} \bar{h}_{j_1 \dots j_k m}^{\alpha} = 0, \quad \text{i.e.,} \quad \sum_{\alpha} h_{i_1 \dots i_k m}^{\alpha} \bar{h}_{j_1 \dots j_k}^{\alpha} = 0.$$

From (2.3) and the second equality of (2.7), we have

$$\begin{aligned} \sum_{\alpha} h_{i_1 \dots i_{k+1}}^{\alpha} \bar{h}_{j_1 \dots j_{k+1}}^{\alpha} \\ = \frac{1}{4^k} \prod_{r=1}^k (\tilde{c} - rc) \sum_{\sigma} \delta_{\sigma(i_1)j_1} \cdots \delta_{\sigma(i_{k+1})j_{k+1}}. \end{aligned}$$

This shows that (2.5) holds for any integer  $k$ .

By the similar argument, we can prove the first equality of (2.5) noting that we may assume  $k > l$  without loss of generality. Q.E.D.

From the second equality of (2.5) we have

$$(2.8) \quad \begin{aligned} \sum_{\alpha} \sum h_{i_1 \dots i_{k+1}}^{\alpha} \bar{h}_{i_1 \dots i_{k+1}}^{\alpha} \\ = \frac{1}{4^k} n(n+1) \cdots (n+k) (\tilde{c} - c)(\tilde{c} - 2c) \cdots (\tilde{c} - kc). \end{aligned}$$

LEMMA 2. If  $c > 0$ , then  $\tilde{c} = vc$  for some positive integer  $v$ .

PROOF. Suppose that there exists no integer  $l$  such that  $\tilde{c} = lc$ . Then, since  $c > 0$  so that  $\tilde{c} > c > 0$ , there exists an integer  $k$  satisfying  $(k-1)c < \tilde{c} < kc$ . For such  $k$ , the right-hand side of (2.8) is negative, which contradicts the fact that the left-hand side is nonnegative. Q.E.D.

Next we prove the following.

LEMMA 3. (1) If  $\tilde{c} = vc$  for a positive integer  $v$ , then  $n + p \geq \binom{n+v}{v} - 1$ .  
 (2) If  $\tilde{c} > 0 \geq c$  or  $0 \geq \tilde{c} > c$ , then  $n + p \geq \binom{n+v}{v} - 1$  for any positive integer  $v$ .

PROOF. Put  $m_l = \binom{n+l}{l} - (n+1)$  and suppose  $p < m_v$ . Let  $H = (H_\mu^\lambda)$  be a matrix of order  $m_v$  defined by

$$\begin{cases} H_\mu^{\alpha-n} = \begin{cases} h_{i_1 i_2}^\alpha & \text{for } \mu \leq m_2, \\ h_{i_1 i_2 i_3}^\alpha & \text{for } m_2 < \mu \leq m_3, \\ \dots & \\ h_{i_1 \dots i_v}^\alpha & \text{for } m_{v-1} < \mu \leq m_v, \end{cases} \\ H_\mu^\lambda = 0 & \text{for } p < \lambda \leq m_v. \end{cases}$$

Then it follows from (2.5) that

$${}^t \bar{H} H = \begin{bmatrix} 2! \frac{\tilde{c}-c}{4} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & * & \\ & & & & \ddots \\ & & & & & \frac{v! \prod_{r=1}^{v-1} (\tilde{c} - rc)}{4^{v-1}} \end{bmatrix},$$

where

$$* = \frac{1}{4^{j-1}} (r_1! + r_2! + \dots + r_l!) \prod_{s=1}^{l-1} (\tilde{c} - sc)$$

and  $r_1 + \dots + r_l = j$ . The assumption on  $c$  and  $\tilde{c}$  implies that the determinant of  $H$  is not zero, which contradicts the definition of the matrix  $H$ . Q.E.D.

Theorem 2 follows immediately from Lemma 3.

In order to complete the proof of Theorem 1, it suffices to show the following.

LEMMA 4. If  $\tilde{c} = vc$  for a positive integer  $v$  and the immersion is full, then  $n + p = \binom{n+v}{v} - 1$ .

PROOF. In consideration of Lemma 3(1), we suppose  $n + p > \binom{n+v}{v} - 1$ .

Let  $A_2 = (A_{2\beta}^\alpha)$  be a matrix of order  $p$  defined by  $A_{2\beta}^\alpha = \Sigma_{k,l} h_{kl}^\alpha \bar{h}_{kl}^\beta$ . It is easily seen that  $A_2$  is a Hermitian matrix. Furthermore, we consider a  $p \times \binom{n+1}{2}$  matrix  $H_2 = (h_{ij}^\alpha)$ . Then these two matrices satisfy

$$A_2 H_2 = \frac{\tilde{c} - c}{2} H_2, \quad H_2 {}^t \bar{H}_2 = A_2,$$

and the matrix  ${}^t \bar{H}_2 H_2$  is nonsingular, which implies that the rank of the matrix  $A_2$  is equal to  $\binom{n+1}{2}$ . We denote by  $N_x$  the normal space to  $M$  at  $x$ , and define a mapping  $f_2$  of  $N_x \times N_x$  into a complex field  $C$  by

$$f_2(X, Y) = \sum_{\alpha, \beta} A_{2\beta}^\alpha \xi_\alpha \bar{\eta}_\beta,$$

where  $X = \Sigma_\alpha \xi_\alpha e_\alpha$  and  $Y = \Sigma_\beta \eta_\beta e_\beta$ . Let  $H_p$  be the set of all Hermitian matrices of order  $p$ , which is considered as a complex vector space. The unitary group  $U(p)$  operates on  $H_p$  as follows: For any Hermitian matrix  $A \in H_p$  and any  $U \in U(p)$ ,  $U(A) = {}^t \bar{U} A U$ .

Since the matrix  $A_2$  is invariant under  $U(p)$ , the mapping  $f_2$  is well defined and it is a positive semidefinite Hermitian form of rank  $r = \binom{n+1}{2}$  so that it can be normalized as

$$f_2(X, X) = \lambda'_1 \xi'_{n+1} \bar{\xi}'_{n+1} + \cdots + \lambda'_r \xi'_{n+r} \bar{\xi}'_{n+r}.$$

This means that we can choose a new unitary frame  $(e_i, e_{\alpha_1}, e_\beta)$  at  $x$  such that

$$(2.9) \quad \omega_i^{\alpha_1} \neq 0, \quad \omega_i^\beta = 0 \quad \text{for} \quad \binom{n+1}{1} \leq \alpha_1 < \binom{n+2}{2}, \beta \geq \binom{n+2}{2}.$$

Similarly, we can choose a new unitary frame  $(e_i, e_{\alpha_1}, e_{\alpha_2}, e_\beta)$  such that

$$(2.10) \quad \begin{aligned} &\omega_{\alpha_1}^{\alpha_2} \neq 0, \quad \omega_{\alpha_1}^\beta = 0 \\ &\text{for} \quad \binom{n+1}{1} \leq \alpha_1 < \binom{n+2}{2}, \\ &\binom{n+2}{2} \leq \alpha_2 < \binom{n+3}{3}, \quad \beta \geq \binom{n+3}{3}. \end{aligned}$$

In fact, we take a unitary frame  $(e_i, e_{\alpha_1}, e_\beta)$  satisfying property (2.9). Let  $A_3 = (A_{3\beta}^\alpha)$  be a matrix of order  $p$  defined by  $A_{3\beta}^\alpha = \Sigma_{i,j,k} h_{ijk}^\alpha \bar{h}_{ijk}^\beta$ . Then  $A_3$  is also a Hermitian matrix of rank  $\binom{n+2}{3}$  and it is invariant under  $U(p)$ . This implies that a mapping  $f_3$  of  $N_x \times N_x$  into  $C$  defined by

$$f_3(X, Y) = \sum_{\alpha, \beta} A_{3\beta}^\alpha \xi_\alpha \bar{\eta}_\beta,$$

where  $X = \sum_{\alpha} \xi_{\alpha} e_{\alpha}$  and  $Y = \sum_{\beta} \eta_{\beta} e_{\beta}$ , is well defined and is also a positive semidefinite Hermitian form of rank  $\binom{n+2}{3}$ . Therefore we can choose a frame satisfying (2.10).

Since  $h_{i_1 \dots i_l}^{\alpha} = 0$  for  $l \geq \nu + 1$ , we can inductively choose a unitary frame  $(e_i, e_{\alpha_1}, \dots, e_{\alpha_{\nu-1}}, e_{\beta})$  at  $x$  such that

$$(2.11) \quad \left\{ \begin{array}{ll} \omega^{\beta} = 0 & \text{for } \beta \geq \binom{n+1}{1}, \\ \omega_i^{\alpha_1} \neq 0, \quad \omega_i^{\beta} = 0 & \text{for } \beta \geq \binom{n+2}{2}, \\ \vdots & \\ \omega_{\alpha_{r-1}}^{\alpha_r} \neq 0, \quad \omega_{\alpha_{r-1}}^{\beta} = 0 & \text{for } \beta \geq \binom{n+r+1}{r+1}, \\ \vdots & \\ \omega_{\alpha_{\nu-2}}^{\alpha_{\nu-1}} \neq 0, \quad \omega_{\alpha_{\nu-2}}^{\beta} = 0 & \text{for } \beta \geq \binom{n+\nu}{\nu}, \\ \omega_{\alpha_{\nu-1}}^{\beta} = 0 & \text{for } \beta \geq \binom{n+\nu}{\nu}, \end{array} \right.$$

where  $\binom{n+r}{r} \leq \alpha_r < \binom{n+r+1}{r+1}$ . Now we consider a distribution  $\mathfrak{M}$  on the frame bundle defined by

$$\omega^{\beta} = 0, \quad \omega_i^{\beta} = 0, \quad \omega_{\alpha_1}^{\beta} = 0, \dots, \omega_{\alpha_{\nu-1}}^{\beta} = 0 \quad \text{for } \beta \geq \binom{n+\nu}{\nu},$$

where  $\binom{n+r}{r} \leq \alpha_r < \binom{n+r+1}{r+1}$ . Then it follows from the structure equations that

$$d\omega^{\beta} = - \sum_{i=1}^n \omega_i^{\beta} \wedge \omega^i - \sum_{r=1}^{\nu-1} \sum_{\alpha_r} \omega_{\alpha_r}^{\beta} \wedge \omega^{\alpha_r} - \sum_{\gamma} \omega_{\gamma}^{\beta} \wedge \omega^{\gamma}$$

$$\equiv 0 \pmod{\omega^{\beta}, \omega_i^{\beta}, \omega_{\alpha_1}^{\beta}, \dots, \omega_{\alpha_{\nu-1}}^{\beta}},$$

$$d\omega_i^{\beta} = - \sum_{j=1}^n \omega_j^{\beta} \wedge \omega_i^j - \sum_{r=1}^{\nu-1} \sum_{\alpha_r} \omega_{\alpha_r}^{\beta} \wedge \omega_i^{\alpha_r} - \sum_{\gamma} \omega_{\gamma}^{\beta} \wedge \omega_i^{\gamma} + \tilde{\Omega}_i^{\beta}$$

$$\equiv 0 \pmod{\omega^{\beta}, \omega_i^{\beta}, \omega_{\alpha_1}^{\beta}, \dots, \omega_{\alpha_{\nu-1}}^{\beta}},$$

$$d\omega_{\alpha_r}^{\beta} = - \sum_{j=1}^n \omega_j^{\beta} \wedge \omega_{\alpha_r}^j - \sum_{s=1}^{\nu-1} \sum_{\alpha_s} \omega_{\alpha_s}^{\beta} \wedge \omega_{\alpha_r}^{\alpha_s} - \sum_{\gamma} \omega_{\gamma}^{\beta} \wedge \omega_{\alpha_r}^{\gamma} + \tilde{\Omega}_{\alpha_r}^{\beta}$$

$$\equiv 0 \pmod{\omega^{\beta}, \omega_i^{\beta}, \omega_{\alpha_1}^{\beta}, \dots, \omega_{\alpha_{\nu-1}}^{\beta}}, \quad \text{where } \gamma \geq \binom{n+\nu}{\nu}.$$

Therefore it is seen that the distribution  $\mathfrak{M}$  is completely integrable.



For any point  $x$  in  $M_n(c)$ , let  $M(x)$  be the maximal integral submanifold of  $\mathfrak{M}$  through  $x$ . Then  $M(x)$  is of  $\{\binom{n+\nu}{\nu} - 1\}$ -dimensional and by the construction it is totally geodesic in  $M_{n+p}(\tilde{c})$ . Thus there exists an  $\{\binom{n+\nu}{\nu} - 1\}$ -dimensional totally geodesic submanifold of  $M_{n+p}(\tilde{c})$  in which the given submanifold  $M_n(c)$  is immersed. This is a contradiction to the assumption that the immersion is full.

This completes the proof.

## BIBLIOGRAPHY

1. E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23. MR 15, 160.
2. K. Ogiue, *On Kaehler immersions*, Canad. J. Math. **24** (1972), 1178–1182. MR 47 #7663.
3. ———, *n-dimensional complex space forms immersed in  $\{n + n(n+1)/2\}$ -dimensional complex space forms*, J. Math. Soc. Japan **24** (1972), 518–526. MR 46 #9907.
4. ———, *Differential geometry of Kaehler submanifolds*, Advances in Math. **13** (1974), 73–114. MR 49 #11444.

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